

# Convergents as approximants in continued fraction expansions of complex numbers with Eisenstein integers

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## 1 Introduction

It is well-known that the convergents defined in terms of the simple continued fraction expansion of a real number are “best approximants”, in the sense that if  $t \in \mathbb{R}$  and  $\{p_n/q_n\}$  is the corresponding sequence of convergents then for any  $n \in \mathbb{N}$  and  $1 \leq q \leq q_n$ , we have  $|qt - p| \geq |q_n t - p_n|$  for all  $p \in \mathbb{Z}$  (see [2], Theorem 181, or [5], Theorem 7.13, for instance). In this paper we shall be concerned with the analogous issue of comparing  $|qz - p|$  with  $|q_n z_n - p_n|$  (see below), for continued fraction expansions of complex numbers. In this respect Hensley [3] considered the continued fractions expansions of  $z \in \mathbb{C}$  in terms of the ring  $\mathfrak{G}$  of Gaussian integers, defined via the nearest integer algorithm, and showed that if  $\{p_n/q_n\}$  is the corresponding sequence of convergents of  $z \in \mathbb{C}$ , then for any  $n \in \mathbb{N}$  and  $p, q \in \mathfrak{G}$ , such that  $1 \leq |q| \leq |q_n|$ , we have  $|qz - p| \geq \frac{1}{5}|q_n z - p_n|$ . Here we prove an analogous result for the ring of Eisenstein integers, namely  $\mathbb{Z}[\omega]$ , where  $\omega$  is a nontrivial cube root of unity.

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**Theorem 1.1.** *Let  $\mathfrak{E}$  be the ring of Eisenstein integers. Let  $z \in \mathbb{C}$  and  $\{a_n\}$  be a continued fraction expansion of  $z$  over  $\mathfrak{E}$  with respect to the nearest integer algorithm and let  $\{p_n/q_n\}$ , be the corresponding sequence of convergents. Then for any  $q \in \mathfrak{E}$  such that  $1 \leq |q| \leq |q_n|$  and any  $p \in \mathfrak{E}$ ,*

$$|qz - p| \geq \frac{1}{2}|q_n z - p_n|.$$

As an application of Theorem 1.1, we obtain the following result analogous to the classical characterisation of badly approximable numbers.

**Definition 1.2.** We say that  $z \in \mathbb{C}$  is *badly approximable* with respect to  $\mathfrak{E}$  if there exists  $\delta > 0$  such that for all  $p, q \in \mathfrak{E}$ ,  $q \neq 0$ ,  $|z - \frac{p}{q}| \geq \delta/|q|^2$ .

**Corollary 1.3.** *Let  $\mathfrak{E}$  be the ring of Eisenstein integers and  $K$  be the quotient field of  $\mathfrak{E}$ . Let  $z \in \mathbb{C} \setminus K$  and  $\{a_n\}_{n=0}^\infty$  be the continued fraction expansion of  $z$  with respect to the nearest integer algorithm on  $\mathfrak{E}$ . Then  $z$  is badly approximable with respect to  $\mathfrak{E}$  if and only if  $\{|a_n|\}_{n=0}^\infty$  is bounded.*

## 2 Preliminaries

Let  $\mathfrak{E}$  be the ring of Eisenstein integers. Then  $\mathfrak{E}$  can also be realised as  $\mathbb{Z}[\rho]$ , where  $\rho = \frac{1}{2} + \frac{i}{2}\sqrt{3}$ , which is a 6th root of unity. We recall that the nearest integer algorithm over  $\mathfrak{E}$  is by defined a map  $f : \mathbb{C} \rightarrow \mathfrak{E}$  such that for all  $z \in \mathbb{C}$ ,  $|z - f(z)| \leq |z - a|$  for all  $a \in \mathfrak{E}$ ; the condition determines  $f(z)$  uniquely for  $z$  in the complement of a countable set of lines, while for other points, which are equidistant from distinct elements of  $\mathfrak{E}$ , there can be multiple choices; and we shall call any  $f$  as above a nearest integer algorithm (we may nevertheless refer to it as “the nearest integer algorithm”, as the multiple choices, for points for which they are available, do not play any role in our results.).

For any  $z \in \mathbb{C}$  we get two sequences  $\{a_n\}_{n=0}^m$  and  $\{z_n\}_{n=0}^m$ , where  $m$  is either a nonnegative integer or  $\infty$ , as follows: we set  $z_0 = z$ , and having defined  $z_0, \dots, z_n$  for some  $n \geq 0$  we set  $z_{n+1} = (z_n - f(z_n))^{-1}$  if  $f(z_n) \neq z_n$ , and terminate the sequence, choosing  $m = n$ , if  $f(z_n) = z_n$ , and define  $a_n = f(z_n)$  for all  $n = 0, \dots, m$ . The sequence  $\{a_n\}_{n=0}^m$  is called the *continued fraction expansion* of  $z$ ,  $a_n$  are called the *partial quotients* of the expansion, and  $\{z_n\}_{n=0}^m$  is called the corresponding *iteration sequence*.

With the continued fraction expansion  $\{a_n\}_{n=0}^m$ , of  $z \in \mathbb{C}$ , we associate two sequences  $\{p_n\}_{n=-1}^\infty$  and  $\{q_n\}_{n=-1}^\infty$  defined recursively by the relations

$$p_{-1} = 1, p_0 = a_0, p_{n+1} = a_{n+1}p_n + p_{n-1}, \text{ for all } n \geq 0, \text{ and}$$

$$q_{-1} = 0, q_0 = 1, q_{n+1} = a_{n+1}q_n + q_{n-1}, \text{ for all } n \geq 0.$$

The pair of sequences  $\{p_n\}_{n=-1}^\infty, \{q_n\}_{n=-1}^\infty$  is called the  $\mathcal{Q}$ -pair corresponding to the expansion. It is known that  $q_n \neq 0$  for all  $n \geq 1$  and if  $m$  is infinite  $p_n/q_n \rightarrow z$  as  $n \rightarrow \infty$ ; see [1], for instance;  $p_n/q_n$  are called the *convergents* of the expansion. We recall here the following result from [1].

**Theorem 2.1.** *Let the notation be as above. Then for all  $n \geq 1$ ,  $|q_{n+1}/q_{n-1}| \geq \frac{3}{2}$ .*

Apart from the notation as above we shall also set, for all  $n \geq 1$ ,  $r_n = q_{n-1}/q_n$ . We note that Theorem 2.1 has the following obvious consequence.

**Corollary 2.2.** *Let the notation be as above. Then for all  $n \geq 1$ , either  $|r_n| \leq \sqrt{\frac{2}{3}}$ , or  $|r_{n+1}| \leq \sqrt{\frac{2}{3}}$ .*

For the proof of Theorem 1.1 we recall also the following standard properties of the sequences associated with the continued fraction expansions.

**Proposition 2.3.** *For all  $n \geq 0$  the following statements hold : i)  $|z_{n+1}| \geq \sqrt{3}$ , ii)  $|q_n| < |q_{n+1}|$ ,*

$$\text{iii) } z = \frac{p_n z_{n+1} + p_{n-1}}{q_n z_{n+1} + q_{n-1}}, \quad \text{iv) } |q_n z - p_n| = \left| \frac{1}{(z_{n+1} q_n + q_{n-1})} \right|.$$

*Proof.* Assertion (i) follows from the fact that  $z_{n+1}^{-1} = (z_n - a_n)$  is contained in the ball of radius  $1/\sqrt{3}$ , in fact in the hexagon with vertices at  $\rho^k i/\sqrt{3}$ ,  $k = 0, \dots, 5$  (see [1], § 5. For Assertion (ii) we refer the reader to [4], or to [1], Theorem 5.1, where it is proved in a general setting. Assertion (iii) and (iv) are general facts about continued fraction expansions, that follow from straightforward manipulations on the recurrence relations; see [1], Proposition 2.1(iii), and the proof of Proposition 2.1(iv), for ready reference).  $\square$

### 3 On the quality of convergents as approximants

Through the section we fix  $z \in \mathbb{C}$  and follow the notation as in the last section associated with the continued fraction expansion of  $z$  with respect to a nearest integer algorithm. We shall now compare  $|qz - p|$  with  $|q_n z - p_n|$  for  $q$  such that  $1 \leq |q| \leq |q_n|$ , and prove Theorem 1.1, and then Corollary 1.3.

*Proof of Theorem 1.1:* Let  $n \in \mathbb{N}$  and  $p, q \in \mathfrak{E}$  be given such that  $1 \leq |q| \leq |q_n|$ . Since  $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$ , it follows that there exist  $\alpha, \beta \in \mathfrak{E}$  such that  $\begin{pmatrix} p \\ q \end{pmatrix} = \alpha \begin{pmatrix} p_n \\ q_n \end{pmatrix} + \beta \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$ ; thus  $p = \alpha p_n + \beta p_{n-1}$  and  $q = \alpha q_n + \beta q_{n-1}$ . Now, by

Proposition 2.3 we have  $z = \frac{z_{n+1} p_n + p_{n-1}}{z_{n+1} q_n + q_{n-1}}$ , and hence

$$|qz - p| = |q| \left| z - \frac{p}{q} \right| = |q| \left| \frac{z_{n+1} p_n + p_{n-1}}{z_{n+1} q_n + q_{n-1}} - \frac{\alpha p_n + \beta p_{n-1}}{\alpha q_n + \beta q_{n-1}} \right| = \left| \frac{\beta z_{n+1} - \alpha}{z_{n+1} q_n + q_{n-1}} \right|,$$

using that  $|p_n q_{n-1} - q_n p_{n-1}| = 1$  and  $\alpha q_n + \beta q_{n-1} = q$ . Also, by Proposition 2.3(iv)

$$|q_n z - p_n| = \left| \frac{1}{z_{n+1} q_n + q_{n-1}} \right|,$$

and hence

$$|qz - p| = |\beta z_{n+1} - \alpha| |q_n z - p_n|.$$

To prove the theorem it therefore suffices to show that  $|\beta z_{n+1} - \alpha| \geq \kappa$  for all  $\kappa < \frac{1}{2}$ . Let  $\kappa < \frac{1}{2}$  be given. We shall suppose that  $|\beta z_{n+1} - \alpha| < \kappa$  and arrive at a contradiction.

First suppose that, if possible,  $\beta = 0$ . Then by the assumption as above  $|\alpha| < \kappa < 1$  and since  $\alpha \in \mathfrak{E}$  we get that  $\alpha = 0$ . But in turn this implies that  $q = \alpha q_n + \beta q_{n+1} = 0$ , contrary to the hypothesis. Hence  $\beta \neq 0$ . Now suppose that  $|\beta| = 1$ . Then replacing  $p$  and  $q$  by their multiples by a fixed unit in  $\mathfrak{E}$  we may assume  $\beta = 1$ . Then we have  $|z_{n+1} - \alpha| < \kappa$ . Since  $\kappa < \frac{1}{2}$  this implies that  $\alpha = a_{n+1}$  and thus  $q = \alpha q_n + \beta q_{n-1} = a_{n+1} q_n + q_{n-1} = q_{n+1}$ , which is a contradiction since by hypothesis  $|q| \leq |q_n| < |q_{n+1}|$ , where the last inequality is as in Proposition 2.3(ii). Thus  $|\beta| > 1$ , and since  $\beta \in \mathfrak{E}$  we get that  $|\beta| \geq \sqrt{3}$ .

Since  $|\alpha q_n + \beta q_{n-1}| = |q| \leq |q_n|$ , dividing by  $|q_n|$  we get  $|\alpha + \beta r_n| \leq 1$ . Since by assumption  $|\beta z_{n+1} - \alpha| < \kappa$ , we get that

$$|\beta| |z_{n+1} + r_n| = |\beta z_{n+1} + \beta r_n| \leq |\beta z_{n+1} - \alpha| + |\alpha + \beta r_n| < \kappa + 1.$$

Recall that by Corollary 2.2 either  $|r_n|$  or  $|r_{n-1}|$  is at most  $\sqrt{\frac{2}{3}}$ . Suppose  $|r_n| \leq \sqrt{\frac{2}{3}}$ . Then, recalling that  $|\beta| \geq \sqrt{3}$  and  $|z_{n+1}| \geq \sqrt{3}$  (cf. Proposition 2.3(i)), we get

$$\kappa + 1 > |\beta||z_{n+1} + r_n| \geq |\beta|(|z_{n+1}| - |r_n|) \geq \sqrt{3} \left( \sqrt{3} - \sqrt{\frac{2}{3}} \right) = 3 - \sqrt{2} > \frac{3}{2},$$

which however contradicts the choice of  $\kappa$ .

Now suppose that  $|r_n| > \sqrt{\frac{2}{3}}$  and  $|r_{n-1}| \leq \sqrt{\frac{2}{3}}$ . We have  $z_{n+1} = 1/(z_n - a_n)$  and  $r_n = 1/(a_n + r_{n-1})$ , and hence

$$|z_{n+1} + r_n| = \left| \frac{1}{z_n - a_n} + \frac{1}{a_n + r_{n-1}} \right| = \frac{|z_n + r_{n-1}|}{|(z_n - a_n)(a_n + r_{n-1})|} = |z_n + r_{n-1}| |z_{n+1}| |r_n|.$$

As  $|\beta| \geq \sqrt{3}$ ,  $|z_n| \geq \sqrt{3}$ ,  $|r_{n-1}| \leq \sqrt{\frac{2}{3}}$ ,  $|z_{n+1}| \geq \sqrt{3}$  and  $|r_n| > \sqrt{\frac{2}{3}}$ , it follows that

$$|\beta||z_{n+1} + r_n| = |\beta||z_n + r_{n-1}| |z_{n+1}| |r_n| \geq \sqrt{3} \left( \sqrt{3} - \sqrt{\frac{2}{3}} \right) \sqrt{3} \sqrt{\frac{2}{3}} > \frac{3}{2}.$$

Thus we get  $\kappa + 1 > \frac{3}{2}$ , again contradicting the choice of  $\kappa$ . Therefore  $|\beta z_{n+1} - \alpha| \geq \kappa$ , as sought to be proved.  $\square$

*Proof of Corollary 1.3:* Since  $|z_{n+1} - a_{n+1}| < 1$  and  $|q_{n-1}| < |q_n|$ , we have

$$|a_{n+1}| - 2 \leq |z_{n+1} + \frac{q_{n-1}}{q_n}| \leq |a_{n+1}| + 2,$$

for all  $n$ . Since, by Proposition 2.3(iv), we have  $|q_n||q_n z - p_n| = (|z_{n+1} + \frac{q_{n-1}}{q_n}|)^{-1}$ , it follows that if  $\{|a_n|\}$  is unbounded then for any  $\delta > 0$  there exists  $n$  such that  $|q_n||q_n z - p_n| < \delta$ , so  $z$  is not badly approximable.

Now suppose that  $\{|a_n|\}$  is bounded, say  $|a_n| \leq M$  for all  $n$ . Then

$$|q_n z - p_n| = |q_n|^{-1} (|z_{n+1} + \frac{q_{n-1}}{q_n}|)^{-1} \geq \frac{1}{(M + 2)|q_n|}$$

for all  $n$ . Now let  $p, q \in \mathfrak{E}$  with  $q \neq 0$  be arbitrary, and let  $n \in \mathbb{N}$  be such that  $|q_{n-1}| \leq |q| \leq |q_n|$ . Since  $q_n = a_n q_{n-1} + q_{n-2}$  and  $|q_{n-2}| < |q_{n-1}|$ , we then have

$|q_n| \leq (M+1)|q_{n-1}| \leq (M+1)|q|$ . By Theorem 1.1 we have  $|qz - p| \geq \frac{1}{2}|q_n z - p_n|$  and hence

$$|q||qz - p| \geq \frac{1}{2}|q||q_n z - p_n| \geq \frac{1}{2} \frac{|q|}{(M+2)|q_n|} \geq \frac{1}{2(M+2)^2},$$

and hence  $|z - \frac{p}{q}| \geq \delta|q|^{-2}$ , with  $\delta = \frac{1}{2}(M+2)^{-2}$ . Since this holds for all  $p, q \in \Gamma$ ,  $q \neq 0$  we get that  $z$  is badly approximable.  $\square$

**Remark 3.1.** An argument as in the proof of Theorem 1.1 does not yield a similar result in the case of the ring  $\mathfrak{G}$  of Gaussian integers (for which the corresponding result is proved in [3] with the constant  $\frac{1}{5}$  in place of  $\frac{1}{2}$ ), since in getting a lower bound for  $\kappa + 1$  as in the proof we would only have at our disposal the estimates  $|z_{n+1}| \geq \sqrt{2}$ , and the resulting bound may be seen to be not good enough.

## References

- [1] S.G. Dani, Continued fraction expansions for complex numbers – a general approach, *Acta Arith.* 171 (2015), 355 - 369.
- [2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Sixth edition, Revised by D. R. Heath-Brown and J. H. Silverman, Oxford University Press, Oxford, 2008.
- [3] D. Hensley, *Continued Fractions*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
- [4] R.B. Lakein, Approximation properties of some complex continued fractions, *Monatshefte für Mathematik* 77 (1973), 396–403.
- [5] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery, *An introduction to the Theory of Numbers*, Fifth edition, John Wiley & Sons, Inc., New York, 1991.

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